Journal of the Korean Physical Society, Vol. 52, No. 2, February 2008, pp. 198∼202

Frequency-Entrainment Measures in Coupled-Oscillator Systems

Hyunsuk HONG
Department of Physics and Research Institute of Physics and Chemistry, Chonbuk National University, Jeonju 561-756

Beom Jun KIM*
Department of Physics, BK21 Physics Research Division, and Institute of Basic Science, Sungkyunkwan University, Suwon 440-746

(Received 14 November 2007)

We study the frequency entrainment behavior in systems of coupled oscillators. In particular, we pay attention to the measures for detecting the frequency entrainment. We consider a system of coupled oscillators in three dimensions (3D) as one example. Whereas the system does not exhibit the phase synchrony at any finite coupling strength in the thermodynamic limit, the frequencies of the oscillators are known to be entrained in 3D. We illustrate several measures that have been used in the previous studies to detect the frequency entrainment and compare their scaling behaviors near criticality. Via a finite-size scaling analysis combined with the recently reported critical exponents $\beta = 0$ and $\nu = 2$ in 3D, we reveal that the Edward-Anderson-type order parameter popularly used in the spin glass systems can, indeed, be used as one good order parameter to detect frequency entrainment.

PACS numbers: 05.45.Xt, 89.75.-k, 05.45.-a
Keywords: Synchronization, Frequency entrainment, Coupled oscillators, Phase transition

I. INTRODUCTION

In recent years, collective synchronization phenomena in a variety of different systems in biology, sociology, and physics, have attracted much attention [1,2]. To capture the basic underlying mechanism of synchronization phenomena in reality, various simplified models have been suggested and analyzed. Among those, the Kuramoto model [2] has been widely explored due to its beautiful simplicity and analytic tractability. The Kuramoto model, in its original form, represents globally coupled limit-cycle nonlinear oscillators. The Kuramoto oscillators with global all-to-all couplings is well known to display phase transition from the desynchronized state to a synchronized one as the coupling strength is increased. In the case of a locally coupled oscillator system, on the other hand, although there already exist some works on it [3–5], understanding of the synchronization transition and its mechanism is still in an early stage. Beyond the importance in theoretical understanding, we believe that also in the application point of view, the synchronization phenomenon in locally coupled systems deserves more intense research because local coupling is much more realistic than global coupling in any actual realization of an oscillator system.

Collective synchronization in systems of coupled oscillators needs to be classified based on two kinds of behaviors: One is phase synchronization in which the phases of the oscillators become identical, and the other is frequency entrainment in which the frequencies or phase velocities are equalized. Frequency entrainment has been also termed as *"phase locking,"* because the phase difference of the oscillators is locked and unchanged in time. It should be noted that oscillators in the phase-locked state (or the frequency-entrained state) do not necessarily possess phase synchrony.

In the present paper, we focus on the frequency-entrainment behavior in a three-dimensional system of locally coupled oscillators. In particular, we illustrate several quantities for detecting frequency entrainment. Through a finite-size scaling analysis, we find that order parameters of the Edward-Anderson type can be used in general to detect a frequency-entrainment transition.

This paper is organized as follows: In Sec. II, we introduce the equations of motion for a system of locally coupled oscillators in three dimensions and present the numerical procedures used. In Sec. III we present several quantities as order parameters to detect frequency entrainment, and in Sec. IV via the finite-size scaling analysis, we pick one that successfully serves as one good order parameter. Finally, we summarize our work and discuss implications in Sec. V.

*Corresponding Author: beomjun@skku.edu

II. MODEL: LOCALLY-COUPLED OSCILLATORS

To explore the frequency entrainment behavior, we now begin with the set of equations of motion governed by

$$\frac{d\varphi_i}{dt} = \omega_i - K \sum_{\langle i,j \rangle} \sin(\varphi_i - \varphi_j), \tag{1}$$

where φ_i represents the phase of the ith limit-cycle oscillator ($i = 1, 2, \cdots, N$) located at sites of the $L \times L \times L$ three-dimensional cubic lattice ($N = L^3$). The term ω_i on the right-hand side in Eq. (1) denotes the natural frequency of the ith oscillator and is assumed to be randomly distributed according to the Gaussian distribution function $g(\omega)$ characterized by the correlation $\langle \omega_i \omega_j \rangle = 2\sigma\delta_{ij}$ and zero mean ($\langle \omega_i \rangle = 0$). The second term on the right-hand side of Eq. (1) describes the local coupling with the sum over the nearest-neighbor pairs without double counting. The positive coupling strength is considered ($K > 0$), so it is assumed to be *ferromagnetic*. In other words, we feel that the neighboring oscillators favor their phase difference being minimized. A negative coupling strength ($K < 0$) corresponds to repulsive coupling.

Let us now look into the system, considering the effects of coupling. When the coupling is absent ($K = 0$) in the system, each limit-cycle oscillator evolves with its own natural frequency ω_i according to $d\varphi_i/dt = \omega_i$, yielding a trivially desynchronized state. On the other hand, when finite coupling ($K > 0$) comes into the system, locally synchronized regions can emerge. The oscillators in the locally synchronized regions evolve with a coupling-modified effective common frequency ω_{eff}, where ω_{eff} can be different from the natural frequency ω_i. The scattered natural frequencies and the coupling compete with each other: the former drives the system toward a desynchronized state whereas the latter causes the system to favor a synchronized state. Accordingly, when the coupling is strong enough to overcome the dispersion of natural frequencies, the system can exhibit collective synchronization.

In existing studies of synchronization of coupled oscillators, the phase synchronization has been mostly explored, which is usually measured by

$$\Delta e^{i\theta} \equiv \frac{1}{N} \sum_{j=1}^{N} e^{i\varphi_j}, \tag{2}$$

where Δ (θ) is the magnitude (the phase) of the complex order parameter for the phase synchronization. When the coupling is too weak the phases of the oscillators are not synchronized, yielding the fully random phase ($\Delta = 0$) in the thermodynamic limit ($N \to \infty$). On the other hand, as the coupling strength increases, the oscillators become synchronized, leading to $\Delta > 0$. Previous studies found that the phase synchronization transition did not occur in systems with dimensions less than four ($d < 4$) [5], which implies $\Delta = 0$ for the whole region of the coupling strength K in the thermodynamic limit. According to the previous studies [5], we found that a 3D system showed a stretched exponential decaying behavior, $\Delta \sim \exp[-(\sigma/4\pi^3 K^2)L]$, in the region of strong coupling. On the other hand, in the region of weak coupling, the phase evolves according to $\varphi_i(t) \approx \varphi_i(0) + \omega_i t$, which yields $\Delta \sim N^{-1/2} = L^{-3/2}$ because the phases $\{\varphi_j\}$ of N oscillators take fully random values.

III. FREQUENCY ENTRAINMENT

We now study the main issue of the present work, frequency entrainment. Even though the 3D system does not exhibit phase synchrony [4, 5], it can still display frequency entrainment at finite couplings. As a simple analogy, one can view frequency-entrained oscillators as people running around a stadium exactly at the same speed: If all people form a localized group, the situation corresponds to a phase-synchronized state. Frequency entrainment without phase synchrony corresponds to the situation in which people, albeit at identical speeds, are spread randomly along the track of the stadium.

Frequency entrainment is known to exist in the system at $d > 2$ [4,5]. In particular, at $d = 3$, the frequency-entrained state is found to appear at a finite coupling strength in the thermodynamic limit. In the sense that a three-dimensional system has broad applications in reality, the exploration of the frequency entrainment behavior in 3D is regarded as a very important one.

In previous studies, the frequency-entrainment behavior was investigated by using several different quantities. For example, in Ref. 4, if two coupled oscillators have almost the same time-averaged phase velocities, they are considered as entrained, and the ratio r of the number N_s of entrained oscillators in the biggest cluster and the total number N of oscillators is used as an order parameter to detect frequency entrainment, *i.e.*,

$$r \equiv \lim_{N \to \infty} \langle N_s/N \rangle. \tag{3}$$

If system-wide frequency entrainment occurs, $r \approx 1$ is measured while if the average phase velocities of coupled oscillators are not correlated with each other, $r \approx 0$ is detected. Although r can measure the degree of frequency entrainment in a successful way, we believe that it is worthwhile to pursue other possible order parameters because the calculation of r takes a longer time than the calculation of other local quantities like Δ in Sec. II. In addition to the calculation time required to compute the average phase velocities, we need to find information on which oscillator belongs to which entrained cluster. Furthermore, the definition of an entrained cluster has an ambiguity because we need to make a somehow artificial distinction on how big a difference in the average

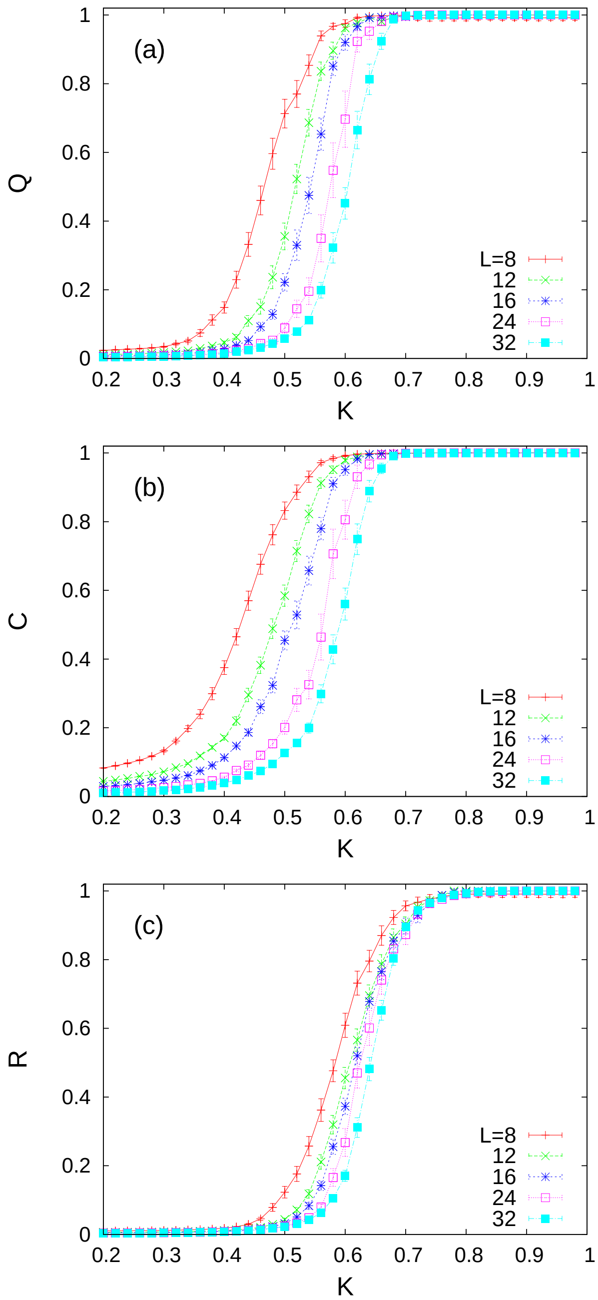

Fig. 1. Frequency-entrainment order parameters (a) Q, (b) C and (c) R versus the coupling strength K at various system sizes. In (b), C is the time average of the autocorrelation function in Eq. (6). All quantities appear to be legitimate as order parameters detecting the frequency entrainment transition, with $m(K \to 0) \to 0$ and $m(K \to \infty) \to 1$ ($m = Q, C$ and R).

phase velocities should be regarded as different. In other words, if two oscillators have a difference δ in their average phase velocities, we need to use a criterion value c such that if $\delta > c$, the two are detected to be different while they are considered to belong to the same entrained cluster otherwise. Accordingly, if we change to a more strict criterion (the smaller c), the oscillators that belonged to identical clusters are now regarded as being in different clusters, which yields a smaller value of r.

In order to remedy such a drawback of using the fraction r, one can instead attempt to use

$$q \equiv \left\langle \frac{1}{N} \sum_{j=1}^{N} e^{i\dot{\varphi}_j} \right\rangle, \tag{4}$$

which is an extension of the order parameter defined in Eq. (2) for phase synchronization. However, although the use of a 2π-periodic function for the phase variable φ_j is naturally justified in Eq. (2), the non-periodicity of the phase velocity $\dot{\varphi}_j$ can hardly make a similar expression for the phase velocity a reasonable choice. In particular, in the region of weak coupling, the so-called *runaway* oscillators exist [5], and the use of q can lead to a peculiar situation that although the runway oscillators have different phase velocities from others, they can still positively contribute to the value of q if the phase velocity happens to differ by a multiple of 2π. In that sense, q is not considered as an appropriate one for the study of frequency entrainment.

We next consider the frequency order parameter Q presented in Ref. 5. Instead of using the geometric information of the connection structure to compute r, one can simplify the calculation by counting only the number of oscillators with the same frequency, neglecting whether or not they are connected. The oscillators are grouped according to their average phase velocities, and the maximum number N_n of oscillators with identical frequency is measured to define [5]

$$Q \equiv \lim_{N\to\infty} \langle N_n/N \rangle. \tag{5}$$

In the system of globally coupled oscillators, the order parameters r and Q are identical to each other. For locally coupled oscillators, on the other hand, Q is always larger than r because the former counts more oscillators as belonging to the same group. Figure 1(a) displays a plot of the frequency order parameter Q versus the coupling strength K for various system sizes. It is found that Q, in sharp contrast to the phase synchronization order parameter Δ [5], saturates to a nonzero value in the strong coupling region, which implies the existence of a transition from a frequency-detrained phase to a frequency-entrained one, as K is increased.

We next consider the frequency-entrainment order parameter written in the form of the autocorrelation function

$$C(t) \equiv \left\langle \left| \frac{1}{N} \sum_{j=1}^{N} \mathrm{e}^{i[\varphi_j(t+t_s)-\varphi_j(t_s)]} \right| \right\rangle, \tag{6}$$

where t_s is a time in the steady state. In the long-time limit $t \to \infty$, one has $[\varphi_j(t+t_s) - \varphi_j(t_s)]/t \approx \bar{\omega}_j$, with the time-averaged phase velocity $\bar{\omega}_j$. In the weak coupling region ($K \ll K_c$), the averaged phase velocity $\bar{\omega}_j$

is randomly distributed, which yields $C(t) \to 0$ in the long-time limit. On the other hand, in the strong coupling region ($K \gg K_c$), most oscillators have the same frequency, yielding $C(t) \to 1$. Accordingly, we expect that the time-average C of the autocorrelation function $C(t)$ plays the role of a good order parameter for frequency entrainment. It is to be noted that the definition of C is almost the same as that of the Edward-Anderson order parameter popularly used in spin glass systems and newly suggested in a coupled oscillator system [3].

In the frequency-entrained phase, a cluster of macroscopic size $O(N)$ with an identical effective frequency $\omega_{\rm eff}$ exists. A phase fluctuation within such a *percolated* cluster is very small in a frame rotating with an angular velocity $\omega_{\rm eff}$. In other words, the phase φ_i of each oscillator in the percolated cluster is written as $\varphi_i \approx \omega_{\rm eff} t + \psi_i$, where the difference between phase fluctuation ψ_i is bounded: $|\psi_i - \psi_j| \ll 1$ for both i and j belonging to the percolated cluster. In Figure 1(b), we display the time-averaged value C of $C(t)$ versus the coupling strength K: C is also shown to play the role of an order parameter for the frequency entrainment transition at $K = K_c \approx 0.70$, which is consistent with the result obtained from Q defined in Eq. (5) [5].

We now suggest another type of the frequency order parameter R defined in terms of the *instantaneous* phase velocity $\dot{\varphi}_i(t)$. We first build a histogram of $\dot{\varphi}_i(t)$ at a given time t after steady state is approached. $R(t)$ is simply defined as the maximum value of the normalized histogram at time t, which is then used to compute the long-time average R. The definition of R is similar to Q in Eq. (5) discussed above except that R is based on the instantaneous phase velocity while Q uses the time-averaged velocity. We believe R to be more stringent order parameter than Q because the instantaneous formation and decomposition of a frequency-entrained cluster cannot be captured by Q. In the frequency-detrained phase, the instantaneous phase velocities of oscillators differ from each other, making $R(t)$ small. In the opposite limit of the frequency-entrained state, on the other hand, most oscillators have almost the same value of the phase velocity, which makes the histogram sharply peaked, yielding $R(t) \approx 1$. Accordingly, as K is increased, we expect to observe a frequency entrainment transition, which is what Figure 1(c) suggests. R again is found to exhibit a transition at $K \approx 0.70$, in a good agreement with the observations based o both Q and C, as shown in Figure 1(a) and (b), respectively.

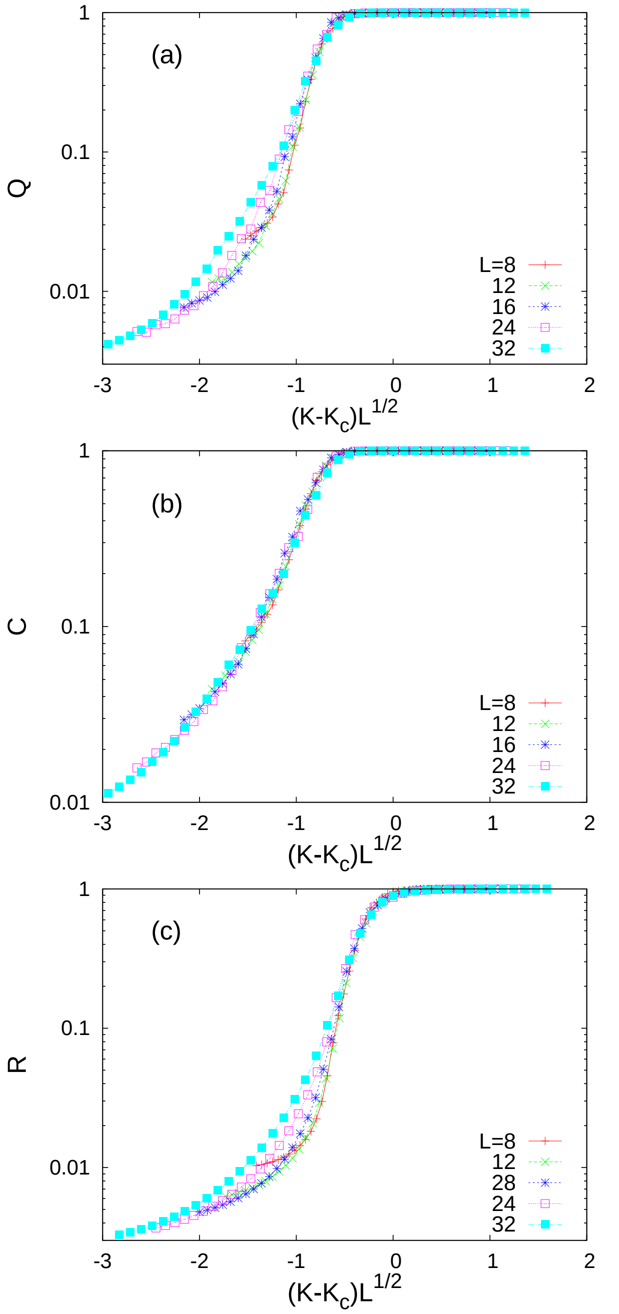

Fig. 2. Finite-size scalings of the frequency-order parameters (a) Q, (b) C and (c) R (compare with Figure 1). The scaling form $m = L^{-\beta/\nu} f \ \ (K - K_c)L^{1/\nu}$ is used [$m = (Q, C, R)$] with the critical exponents $\beta = 0$, $\nu = 2$. Only K_c is tuned to yield the best scaling collapse of data points. In (a) and (b), $K_c = 0.74$ is used while $K_c = 0.70$ in (c).

IV. FINITE-SIZE SCALING

We have shown in Sec. III that the described order parameters Q, C and R can be used to detect the frequency-entrainment transition in locally-coupled oscillators in three dimensions. In detail, all the quantities, Q, C and R, unanimously decay to zero as $K \to 0$ and approach unity beyond some critical value around $K \approx 0.7$. We would like to point out that there is no *a priori* reason to say one is better than the others. We, in this section, use a finite-size scaling analysis to find out which order parameter among Q, C, and R proposed in Sec. III is consistent with the critical scaling. The finite-size scaling form of the order parameter reads as

$$m = L^{-\beta/\nu} f\big((K - K_c)L^{1/\nu}\big), \quad (7)$$

where $m = (Q, C, R)$ is the frequency-entrainment order parameter and K_c is the critical coupling strength splitting the detrained phase from the entrained one. Via a thorough analysis of the dependence of the behavior of the frequency-entrainment order parameter on the coupling strength K, a sharper curve is found as the system size L increases [3,5]. This numerical observation allows us to expect a sudden jump at K_c in the thermodynamic limit, which implies $\beta = 0$ [3]. A completely different feature in the probability distribution function of average phase velocity between a globally coupled system and a low-dimensional one such as $d = 3$ also strongly supports this [3,5]. Also, from some intuition about the factors governing the typical domain size, the correlation length exponent $\nu = 2/(d - 2)$ is obtained [3]. Such an estimate is obtained from the idea that the δK needed for merging the neighboring synchronized clusters with the phase difference $\Delta\phi_0 \sim \xi^{(4-d)/2}$ offsets the phase gradient $\Delta\phi_0/\xi \sim \xi^{(2-d)/2}$ [3].

In the present paper, we check how well various order parameters for frequency entrainment satisfy finite-size scaling with the exponents $\beta = 0$ and $\nu = 2$ proposed in Ref. 3. For each suggested order parameter, only K_c in the scaling form is tuned to make a better collapse of all data obtained for different system sizes $L = 8, 12, 16, 24$ and 32. Figure 2 shows the scaling plots for quantities Q, C and R. All Q, C and R are found to show a good scaling near K_c, although some failure of the scaling collapse far away K_c is shown for larger sizes for Q and R. We note that C among those various frequency order parameters appears to satisfy best the finite-size scaling form in Eq. (7).

V. SUMMARY

In summary, we have studied the frequency entrainment of locally coupled oscillators in three dimensions. Frequency entrainment at a finite coupling strength has been detected via three described order parameters, which unanimously exhibit the expected behavior, *i.e.*, vanishing at null coupling and approaching unity in the strong-coupling limit. A finite-size scaling analysis combined with the recently proposed critical exponents was used to answer the question of which among the suggested order parameters can be considered as the best order parameter. Revealed is that the quantity C, defined as the time average of the absolute value of an autocorrelation function, can be used to detect the frequency-entrainment transition. We believe that the identification of the order parameter for the frequency-entrainment transition in this work can contribute much in similar studies, particularly when the critical exponents are sought by using a finite-size scaling analysis.

ACKNOWLEDGMENTS

This work was supported by a Korea Research Foundation grant funded by the Korean Government (Ministry of Education and Human Resource Development) under grant No. KRF-2006-331-C00123 (H. H.) and Grant No. KRF-2006-312-C00548 (B. J. K.).

REFERENCES

[1] For a list of references, see A. T. Winfree, *The Geometry of Biological Time* (Springer-Verlag, New York, 1980); J. Theor. Biol. **16**, 15 (1967); G. B. Ermentrout and J. Rinzel, J. Math. Biol. **22** 1 (1985); H. Daido, Prog. Theor. Phys. **75**, 1460 (1986); **77**, 622 (1987); H. Hong, M. Y. Choi, J. Yi and K.-S. Soh, Phys. Rev. E **59**, 353 (1999); H. Hong, M. Y. Choi, B.-G. Yoon, K. Park and K.-S. Soh, J. Phys. A **32**, L9 (1999); Y. Kuramoto and Y. Kawamura, J. Korean Phys. Soc. **50**, 170 (2007); S. K. Baek, J. Korean Phys. Soc. **50**, 320 (2007); S.-G. Lee, C.-W. Shin and S. Kim, J. Korean Phys. **48**, S192 (2006); S.-G. Lee, S. Kim and S. Tanaka, J. Korean Phys. **46**, L741 (2005); S. H. Strogatz, Physica D **143**, 1 (2000).

[2] Y. Kuramoto, in *Proceedings of the International Symposium on Mathematical Problems in Theoretical Physics*, edited by H. Araki (Springer-Verlag, New York, 1975); Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer-Verlag, Berlin, 1984); Y. Kuramoto and I. Nishikawa, J. Stat. Phys. **49**, 569 (1987).

[3] H. Hong, H. Chaté, H. Park and L.-H. Tang, Phys. Rev. Lett. **99**, 184101 (2007).

[4] H. Sakaguchi, S. Shinomoto and Y. Kuramoto, Prog. Theor. Phys. **77**, 1005 (1987); H. Sakaguchi, *ibid.* **79**, 39 (1988); S. H. Strogatz and R. E. Mirollo, Physica D **31**, 143 (1988); J. Phys. A **21**, L699 (1988); H. Daido, Phys. Rev. Lett. **61**, 231 (1988); T. Aoyagi and Y. Kuramoto, Phys. Lett. A **155**, 410 (1991); M. Bahiana and M. S. C. Massunaga, Phys. Rev. E **49**, R3558 (1994).

[5] H. Hong, H. Park and M. Y. Choi, Phys. Rev. E **70**, 045204(R) (2004); Phys. Rev. E **72**, 036217 (2005).