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Coarse Graining of Complex Networks

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For complex networks geographically embedded in two dimensions (2D), a numerical scheme of coarse graining, analogous to the Kadanoff-type block spin renormalization-group (RG) method in statistical physics is proposed. The geographical coarse-graining method is then applied for a 2D scale-free network model to yield the conclusion that the degree distribution does not change when coarse graining is repeated, reflecting the fact that the scale-free network is a fixed point in the RG sense.

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The recent burst of research activity involving complex network in various disciplines of the sciences has revealed many new and interesting features [1–3]. For example, the neuronal network of *Caenorhabditis elegans* [2], composed of about 300 neuron cells and 14 synaptic couplings per neuron, has been obtained by biologists through direct observations. Subsequent analyses showed that this network has both a small-world property and a high level of clustering [2]. In comparison to the *C. elegans* neural network, the complexity of the human brain is gigantic: It contains about 10^{11} neuron cells, each of which is connected to $10^3 - 10^4$ other neurons via synaptic couplings. Construction of a detailed map of all the neuron connections in the human brain is beyond imagination and will remain so in the future. In the viewpoint of the universality class in statistical physics, understanding the qualitative collective behavior of the brain may not require a detailed microscopic map of the interneuron connections. In this regard, the recent study by Eguíluz *et al.* in Ref. 4 draws much interest: Brain activity was measured from $32 \times 64 \times 64$ sites (called voxels), and an intervoxel correlation was used to map out the functional network of the human brain. Although the number of voxels in Ref. 4 is more than a million, each voxel still contains $O(10^5)$ neuron cells. Consequently, one can say that the brain functional network in Ref. 4 is based on *heavily coarse-grained* information; thus, it is not clear whether the observed scale-freeness of the network is a genuine emerging property of actual interneuron connections or not.

In this paper, we propose a geographic coarse-graining method, which is then applied for the scale-free network model embedded in 2D [5]. As the coarse-graining process is repeated, the degree exponent γ defined by the power-law degree distribution $p(k) \sim k^{-\gamma}$ is shown to be unchanged, implying that the initial network does not have a characteristic length scale. Our result suggests that the scale-free feature of the human brain functional network may not be an artifact of the coarse graining; thus, an increase (or a decrease) on the voxel size is not expected to change the scale-freeness found in Ref. 4.

We first build the geographically embedded scale-free network in Ref. 5: $N = L \times L$ vertices are put on the lattice points of the 2D square net. Then, the edges are placed to satisfy $p(k) \propto k^{-\gamma}$ with degree k (See Ref. 5 for details). Once the network is constructed, we start the following geographic coarse-graining, which is in parallel to the Kadanoff block spin renormalization group procedure in standard statistical mechanical systems (see Fig. 1): The four vertices on each square box of size 2×2 are merged to a single vertex; accordingly, the edges connecting intra-box vertices (dashed lines in Fig. 1) are disregarded, but the inter-box connections are kept (thin solid line in Fig. 1). We also keep track of how strong the edges are by assigning a weight w_{vw} that is simply the number of edges connecting two merged vertices v and w. For example, in Fig. 1, there are two edges (thin solid line) connecting the two square boxes before the merging, which gives rise to a weight $w = 2$ for the edge (thick solid line) connecting the two vertices after the merging.

If we keep all the edges in the coarse-grained network, the average degree increases as the procedure is iterated, eventually resulting in a fully-connected network. To remedy this, we fix the average degree at each step of coarse graining by removing weaker edges with smaller values of the weight. Suppose that we have to remove M_r edges to keep the average degree the same and that there are M_w edges of the weight w. For example, for $M_r < M_{w=1}$, randomly picked M_r edges of $w = 1$ are removed.

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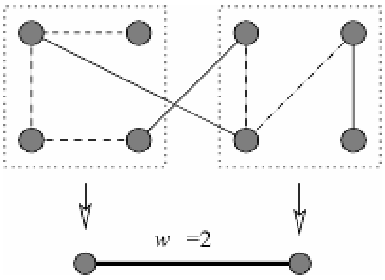

Fig. 1. Coarse-graining procedure. Four vertices in each 2×2 box are merged to a single vertex. After merging, two edges (thin solid lines) connecting vertices in different boxes becomes one edge (thick solid line) with a weight $w = 2$. This procedure is repeated for all 2×2 square boxes, resulting in a coarse-grained network which is four times smaller than the original network. We then remove edges of smaller weights in order to fix the average degree.

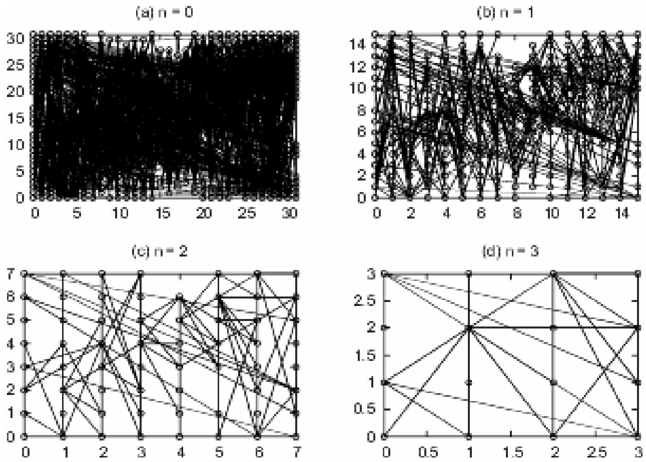

Fig. 2. Geographically embedded scale-free network at the n-th step of coarse graining for (a) $n = 0$, (b) $n = 1$, (c) $n = 2$, and (d) $n = 3$. The initial network ($n = 0$) of size 32×32 in (a) is coarse grained to networks of sizes (b) 16×16, (c) 8×8, and (d) 4×4.

If $M_{w=1} < M_r < M_{w=2}$, all $w = 1$ edges are removed, and $M_r - M_{w=1}$ edges with $w = 2$ are randomly deleted . The above procedure makes sense since in real situations, it is common for coarse graining to be often accompanied by a change in the sensitivity of the measurement: When the system is looked at from a far distance, we only have interest in large-scale structures.

Figure 2 shows the structure of the scale-free networks in a 2D plane after n steps of coarse graining with (a) $n = 0$, (b) $n = 1$, (c) $n = 2$, and (d) $n = 3$. The symbols denote the positions of the vertices in the 2D plane, and the lines denote the edges connecting vertices. An initial scale-free network of size 32×32 corresponding to (a) $n = 0$ with the degree exponent $\gamma = 3$ is built, and the coarse-graining process is iterated to yield smaller networks. The connection structures in Fig. 2 show that the qualitative global feature of the network connection remains the same upon coarse graining: For example, the relative positions of the hub vertices do not change much. From the above observation, one can expect important network properties of the geographically embedded scale-free network in Ref. 5 to not change significantly although the coarse-graining process is repeated. This is of a practical use because the networks to be analyzed often have sizes beyond the computer memory capacity; thus, it is desirable to subtract smaller networks but without loss of important network properties.

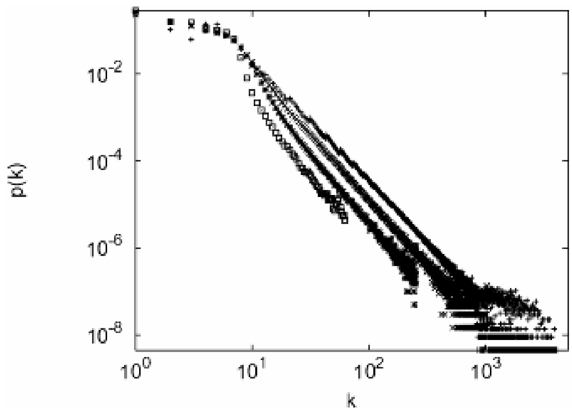

Fig. 3. Degree distribution $p(k)$ versus the degree k for networks with the degree exponent $\gamma = 3$. The original network of size 64×64 is coarse grained three times (from top to bottom). Clearly shown is that the degree distribution remains scale-free with the same degree exponent $\gamma = 3$.

In order to make the above expectation more concrete, the degree distribution at each coarse-graining step is measured and then averaged over 10^4 different network realizations (see Fig. 3). The initial network has size 64×64 with $\gamma = 3$. As the coarse graining is repeated, the degree exponent $\gamma = 3$ is shown to be unchanged, which implies that the initial scale-free network has neither a degree scale nor a length scale. In the viewpoint of the renormalization group formalism in statistical physics, the above finding also suggests that the geographically embedded scale-free network in Ref. 5 is a renormalization group fixed point. We also verified the above conclusion for different values of the degree exponent and confirmed its validity.

The scale-free degree distribution detected in the human brain's *functional* network [4] does not actually imply that the neural network of the human brain is scale free. One reason is the technique in Ref. 4 only measures the functionality correlation of two separate voxels, not the actual path of voxels through which the biochemical signal is transferred. One can also argue that since each voxel contains a large number of neurons [about $O(10^5)$], the observed scale-free distributions can be artifacts of the coarse graining, considering that the recent study in Ref. [6] showed that scale-free distributions can emerge due to merging. Our main result in the present study implies that this is not the case and that, accordingly, the

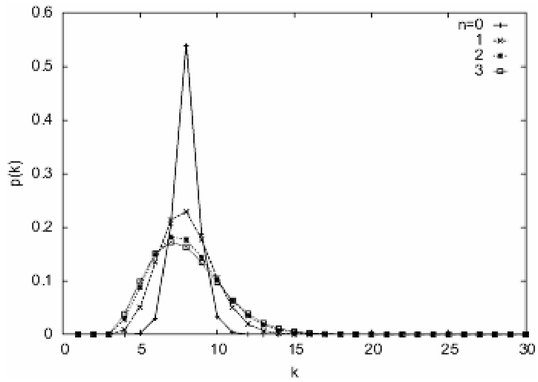

Fig. 4. Degree distribution $p(k)$ at the iteration steps $n =$ 0, 1, 2, and 3 for the geographically embedded WS network at the rewiring probability $P = 0.1$.

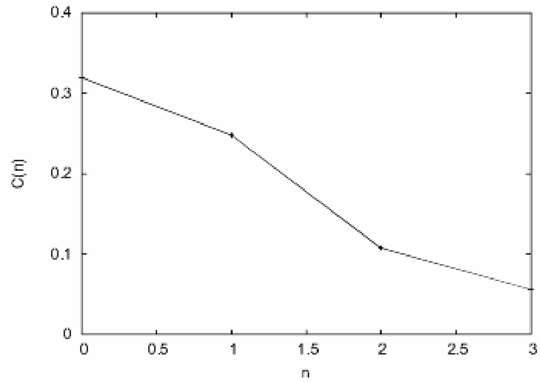

Fig. 5. Clustering coefficient $C(n)$ at the nth iteration step. The geographically embedded WS network of the size $L \times L$ with $L = 128$ at a rewiring probability $P = 0.1$ was used as the initial network before coarse graining.

scale-free distribution in the human brain's functional network is expected to be a genuine property of the brain, not an artifact of coarse-grained information.

We finally study the 2D Watts-Strogatz (WS) network, built similarly to Ref. 2: Vertices are put on the two-dimensional square lattice points, and every vertex is connected to its nearest and next-nearest neighbor vertices. Each edge is visited once, and with the rewiring probability P, is rewired to a randomly chosen vertex. The resulting network is a so-called exponential network because the tail in the degree distribution is exponentially small. We then iterate our geographic coarse-graining procedure with the average degree kept constant at each iteration. In Fig. 4, an initial network of size 128×128 with a rewiring probability $P = 0.1$ is coarse grained n times. As n becomes larger, the degree distribution remains exponential and tends to saturate. In Fig. 5, initial two-dimensional WS networks of size 128×128 with $P = 0.1$ are coarse grained, and the clustering coefficient $C(n)$ (see Ref. 2 for the definition) is plotted as a function of the number, n, of iterations. As the coarse-graining process proceeds, the clustering coefficient is shown to decrease towards zero, which indicates that the RG stable fixed point of the WS network is close to the random network of Erdös and Rényi [7].

In summary, in this work we have proposed a geographical coarse-graining method applicable to networks defined on real spaces. Many real networks, such as the World-Wide Web, the actor collaboration network, the scientific citation network [1], and the network of protein interactions [8], do not possess any geographical meaning. In contrast, the network of Internet servers, power grids, and the network of flight connections are defined on the 2D surface of the earth; thus, each vertex occupies a specific geographic location. By using the scale-free network model in Ref. 5, we showed that a certain type of geographically embedded scale-free network does not change the important network properties, like the scale-free degree distribution, although the coarse-graining process is iterated. This conclusion is then interpreted as an implication of the robustness of the scale-freeness of the recently studied human brain's functional network against changes in the voxel sizes. One can also use the proposed coarse-graining method to subtract a subnetwork from the original network without destroying important network properties. This is to be compared with the recent study of subtracting the tree structure from the original network by using edge betweenness [9]. Both the coarse-graining method in the present work and the subtracting tree structure in Ref. 9 are practically useful in simplifying the original network structures.

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REFERENCES

[1] For general reviews, see, *e.g.*, D. J. Watts, *Small Worlds* (Princeton University Press, Princeton, 1999); R. Albert and A-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002); S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks* (Oxford University Press, New York, 2003).

[2] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).

[3] A-L. Barabási and R. Albert, Science **286**, 509 (1999); A-L. Barabási, R. Albert and H. Jeong, Physica A **272**, 173 (1999).

[4] V. M. Eguíluz, D. R. Chialvo, G. A. Cecchi, M. Baliki and A. V. Apkarian, Phys. Rev. Lett. **94**, 018102 (2005).

[5] A. F. Rozenfeld, R. Cohen, D. ben-Avraham and S. Havlin, Phys. Rev. Lett. **89**, 218701 (2002); D. ben-Avraham, A. F. Rozenfeld, R. Cohen and S. Havlin, Physica A **330**, 107 (2003).
[6] B. J. Kim, A. Trusina, P. Minnhagen and K. Sneppen, Eur. Phys. J. B (in press).
[7] P. Erdös and A. Rényi, Publ. Math. Inst. Hung. Acad. Sci. **5**, 17 (1960).
[8] K.-I. Goh, B. Kahng and D. Kim, J. Korean Phys. Soc. **46**, 551 (2005).
[9] D-H. Kim and H. Jeong, J. Korean Phys. Soc. **44**, 624 (2004).