Quantum coherence and duality in Josephson junctions

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(Received 22 January 1992; revised manuscript received 9 November 1992)

The contribution to the partition function of the instanton tunneling between degenerate vacua is investigated for systems of Josephson junctions with gauge charge induced by external sources. Effects of weak dissipation both due to quasiparticle tunneling and due to shunt resistance are considered via a perturbative method for one- and two-junction cases. The tunneling partition functions display oscillating behavior as functions of the gauge charge, generating oscillating voltages in the presence of external currents. The duality relation in the system is also discussed: In particular, the strong-coupling limit and the weak-coupling limit of a ring of N junctions map onto each other.

In quantum mechanics, the electromagnetic potential plays a fundamental role even in the region of no electromagnetic field, as manifested by the Aharonov-Bohm effect.[1] Thus the energy levels as well as the partition function of a particle moving on a circle are periodic functions of the magnetic flux enclosed by the circle, leading to the interesting possibility of a persistent current in the system.[2] This is to be compared with a Josephson junction, the Hamiltonian of which can be written in the same form as that of a particle on a circle in a cosine-type periodic potential. In the case of a Josephson-junction system, externally induced gauge charge plays the role of magnetic flux. Then the voltage across the junction is a periodic function of the gauge charge, which implies that a persistent voltage drop can be induced without an accompanying dc current.[3] The relevant dynamic variables in such a Josephson-junction system are the phases of the superconducting order parameters, which represent collective degrees of freedom. In this respect, the Josephson-junction system can be considered to display quantum mechanics on a macroscopic scale, in particular, the macroscopic quantum coherence (MQC), which involves coherent tunneling between two or more degenerate vacua. Such effects have been investigated in charge-density-wave systems to reveal oscillations with period $hc/2eN$ for a ring of N correlated chains.[4]

In general, a macroscopic system cannot be sufficiently decoupled from the outside world, and there exist effects of environment. Such dissipation effects on Josephson junctions, which are usually studied with either the Ohmic dissipation due to normal current flow through a shunt resistor or the dissipation due to discrete tunneling of quasiparticles, can be treated by the effective action. The effective dissipative action may be calculated either by integrating out the degrees of freedom of the environment represented by a bath of harmonic oscillators[5] or from the microscopic model Hamiltonian, with macroscopic variables introduced via the Hubbard-Stratonovich transformation.[6] Effects of dissipation on the Aharonov-Bohm oscillation have been considered only for a *free particle* on a circle.[7]

In this paper, we use the instanton formalism[8] to compute the partition functions for systems of Josephson junctions in the presence of dissipation. Effects of dissipation both due to quasiparticle tunneling and due to shunt resistance are considered in the case of single-junction and two-junction systems. In particular, a ring of N junctions is considered at low temperatures, which may be mapped onto a particle on a circle. This leads to an interesting duality relation in the ring between the strong-coupling and weak-coupling limits.

We first consider a single Josephson junction of capacitance C and Josephson coupling energy V_0. The Hamiltonian of the system is given by

$$H=(1/2C)(q-Q)^2+V_0(1-\cos\phi)\ , \tag{1}$$

where ϕ is the phase difference between the two superconducting order parameters, q measures the excess charge of Cooper pairs on the junction, and Q is a gauge charge induced by an applied voltage V or current I. With $C/4e^2\equiv mR^2/\hbar^2\equiv M$ and $Q/2e\equiv f$, Eq. (1) can be written in the form

$$H=(1/2M)(p-f)^2+V_0(1-\cos\phi)\ , \tag{2}$$

which is the Hamiltonian of a particle of mass m and charge $-e$ moving on a circle of radius R in the periodic potential $V_0(1-\cos\phi)$. In this interpretation, ϕ is the azimuthal angle, p is the canonical (angular) momentum conjugate to ϕ, and $f\equiv\Phi/\Phi_0$ is the flux enclosed by the circle in units of the flux quantum $\Phi_0\equiv hc/e$. (Henceforth we set $\hbar\equiv1$.) The Euclidean Lagrangian corresponding to the Hamiltonian equation (2) is given by

$$L_E=\frac{1}{2}M\left[\frac{d\phi}{d\tau}\right]^2+V_0(1-\cos\phi)-if\frac{d\phi}{d\tau}\ , \tag{3}$$

with imaginary time $\tau\equiv it$, which has been also considered in the context of charge-density-wave systems.[4] From the Euclidean action $S_E\equiv\int d\tau L_E$, we obtain the classical solutions of $\delta S_E=0$ with the boundary condition $\phi(\beta/2)-\phi(-\beta/2)=2\pi n$, where $\beta\equiv1/k_BT$ is the inverse temperature and n is the winding number. For $n=1$, such an instanton solution can be calculated exactly in the form of elliptic integrals,[4] and leads to the partition function in the high-temperature limit ($\beta\omega \ll 1$),

$$Z \approx e^{-\beta V_0} Z_{\rm free} ,$$

where $Z_{\rm free}$ is the partition function of a free particle given by the Jacobi theta function

$$Z_{\rm free} = \vartheta_3(f, e^{-2\pi^2 M/\beta}) \equiv \sum_{n=-\infty}^{\infty} \exp\left[-\frac{2\pi^2 M}{\beta} n^2\right] e^{i2\pi n f} . \quad (4)$$

At low temperatures ($\beta\omega \gg 1$), on the other hand, the instanton solution reduces to

$$\phi(\tau) = 4\tan^{-1}[\exp(\pm\omega\tau)] , \quad (5)$$

where the positive (negative) sign represents an instanton (anti-instanton). The solution for the arbitrary winding number n can be constructed by inserting n_1 instantons and $n_2 \equiv n_1 - n$ anti-instantons. This dilute instanton approximation is valid at low temperatures, where the separation between instantons is sufficiently larger than the size of each instanton. It is then straightforward to compute the partition function

$$Z = \sqrt{\omega/\pi} e^{-\beta\omega/2} \sum_{n=-\infty}^{\infty} I_n(\beta\varepsilon_0) e^{2\pi i n f} , \quad (6)$$

where $\varepsilon_0 \equiv 8\sqrt{V_0 \omega/\pi} \exp(-8V_0/\omega)$ and $I_n(x)$ is the modified Bessel function. Equation (6) manifests the periodicity of the partition function in f with period unity. Thus in the presence of an applied direct current I, the partition function displays an oscillation with period $2e/I$. Unless the Josephson energy V_0 and the charging energy $E_C \equiv 2e^2/C$ are comparable to each other, ε_0 is much smaller than ω. In this *macroscopic limit*, we have $\beta\varepsilon_0 \ll 1$ together with $\beta\omega \gg 1$, for which the partition function in Eq. (6) becomes, except for the phase factor,

$$Z \approx 1 + \beta\varepsilon_0 \cos 2\pi f .$$

We now consider the effects of dissipation. In the presence of dissipation, the Euclidean action consists of two parts: $S_E = S_0 + S_D$, where S_0 is the action in the absence of dissipation and S_D represents the dissipative action. In the case of dissipation due to quasiparticle tunneling, S_D can be chosen as[6,7]

$$S_D = \frac{\eta_q}{2\pi} \int_{-\beta/2}^{\beta/2} \int_{-\beta/2}^{\beta/2} d\tau\, d\tau' \frac{\sin^2\{[\phi(\tau') - \phi(\tau)]/4\}}{(\beta/\pi)^2 \sin^2[\pi(\tau'-\tau)/\beta]} , \quad (7)$$

where η_q represents the strength of dissipation. For Ohmic dissipation, on the other hand, the dissipative action takes the form[6]

$$S_D = \frac{\eta_s}{2\pi} \int_{-\beta/2}^{\beta/2} \int_{-\beta/2}^{\beta/2} d\tau\, d\tau' \frac{[\phi(\tau') - \phi(\tau)]^2}{(\beta/\pi)^2 \sin^2[\pi(\tau'-\tau)/\beta]} . \quad (8)$$

For weak dissipation ($\eta_{q,s} \ll 1$), we may use the nondissipative solution given by Eq. (5) to calculate S_D in a perturbative manner.

We first consider dissipation due to quasiparticle tunneling. S_D given by Eq. (7) is divergent for odd winding numbers, and only terms of even winding numbers contribute to the partition function. Therefore we have the one-instanton action $S_E = S_0 + S_D'/2$, where S_D' is the dissipative action for two instantons given by

$$S_D' \approx 1.74\eta_q - (2/\pi)\eta_q \ln(\pi^2/\beta\omega)$$

in the low-temperature limit. For small $\beta\varepsilon_0$, this leads to the partition function

$$Z \approx 1 + \tfrac{1}{2}(\beta\varepsilon \cos 2\pi f)^2 ,$$

where $\varepsilon \approx \varepsilon_0 (\pi^2/\beta\omega)^{\eta_q/\pi} \exp(-0.87\eta_q)$. Therefore the partition function is periodic in f with period 1/2. Weak dissipation due to quasiparticle tunneling reduces not only the amplitude of the oscillating part in the partition function but also the period of the partition function by a factor of 2. This reduction of the amplitude is consistent with the result of Ref. 5, where weak dissipation was shown to reduce the WKB tunneling amplitude by the factor $\exp(-A\eta)$, with A being a number of $O(1)$.

We next consider Ohmic dissipation due to shunt resistance. The dissipative action S_D given by Eq. (8) is divergent for nonzero winding numbers. In the low-temperature limit, it is straightforward to evaluate S_D for a pair of instantons and anti-instantons:

$$S_D \approx 4\pi\eta_s [3 - 2\ln(\pi^2/\beta\omega)] ,$$

which leads to the partition function for small $\beta\varepsilon_0$,

$$Z \approx 1 + \tfrac{1}{4}\beta^2\varepsilon^2 ,$$

with $\varepsilon \approx \varepsilon_0 (\pi^2/\beta\omega)^{4\pi\eta_s} \exp(-6\pi\eta_s)$. Thus Ohmic dissipation again reduces the tunneling amplitude but it destroys the periodicity in f due to the nonperiodic nature of Ohmic dissipation described by Eq. (8).

Heretofore we have considered a single-junction system. We now consider a two-junction system (a ring of two junctions) with the Lagrangian

$$L = \tfrac{1}{2}M(\dot\phi_1^2 + \dot\phi_2^2) + f(\dot\phi_1 + \dot\phi_2) - V_0[1 - \cos(\phi_1 - \phi_2)] , \quad (9)$$

where ϕ_1 and ϕ_2 are the phases of the order parameters of the two superconducting islands. The corresponding Euclidean action can be written in the decoupled form

$$S_E = \int_{-\beta/2}^{\beta/2} d\tau \left[\frac{1}{4}M\left[\frac{d\phi}{d\tau}\right]^2 - if\frac{d\phi}{d\tau}\right] + \int_{-\beta/2}^{\beta/2} d\tau \left[\frac{1}{4}M\left[\frac{d\theta}{d\tau}\right]^2 + V_0(1-\cos\theta)\right] , \quad (10)$$

where we have defined $\phi \equiv \phi_1 + \phi_2$ and $\theta \equiv \phi_1 - \phi_2$ with the boundary conditions

$$\begin{cases} \phi(\beta/2) - \phi(-\beta/2) = 2\pi(m+n) \\ \theta(\beta/2) - \theta(-\beta/2) = 2\pi(m-n) . \end{cases}$$

Thus the partition function is given by

$$Z=\sum_{m,n=-\infty}^{\infty} e^{-\pi^2M(m+n)^2/\beta} I_{m-n}(\beta\varepsilon_0)e^{2\pi i(m+n)f} , \quad (11)$$

where $\varepsilon_0 \equiv 8\sqrt{V_0\omega/\pi}\exp(-8V_0/\omega)$ with $\omega^2\equiv 2V_0/M$. For small $\beta\varepsilon_0$, we have

$$Z\approx\vartheta_3(2f,e^{-4\pi^2M/\beta})+\beta\varepsilon_0 e^{-\pi^2M/\beta} \times \sum_m e^{-4\pi^2Mm^2/\beta}e^{4\pi i f m}\cosh\left[\frac{4M\pi^2}{\beta}m-2\pi i f\right] ,$$

where the first and second terms are periodic in f with periods 1/2 and 1, respectively. In the macroscopic limit ($\beta\varepsilon_0 \to 0$), only the first term survives as pointed out in Ref. 4 by qualitative consideration of the Hamiltonian for the charge-density-wave system. In the presence of dissipation, the partition function can be computed to give

$$Z=\vartheta_3(2f,e^{-4\pi^2M/\beta})[1+\tfrac{1}{2}(\beta\varepsilon)^2] ,$$

where $\varepsilon\approx\varepsilon_0(\pi^2/\beta\omega)^{\eta_q/\pi}e^{-0.87\eta_q}$ for dissipation due to quasiparticle tunneling and $\varepsilon\approx\varepsilon_0(\pi^2/\beta\omega)^{4\pi\eta_s}e^{-6\pi\eta_s}/\sqrt{2}$ for Ohmic dissipation. It is again periodic with period 1/2. Here the preservation of the periodicity in the two-junction system in spite of Ohmic dissipation can be understood as follows. In Eq. (10), the center-of-mass coordinate ϕ and the relative coordinate θ are decoupled from each other, and the gauge charge f is involved only in the center-of-mass part which takes the form of the free-particle action. In such a system it is known that dissipation hardly changes the partition function.[7] Since in the macroscopic limit the dominant contribution comes from the center-of-mass part, the effect of dissipation in the two-junction system is negligible compared with the single-junction case, thus maintaining the periodicity.

We next consider a ring of N Josephson junctions, which, in the absence of dissipation, has been considered via a variational method,[9] and generalize the Lagrangian in Eq. (9):

$$L=\frac{1}{2}M\sum_{k=1}^{N}\dot\phi_k^2+f\sum_{k=1}^{N}\dot\phi_k -V_0\sum_{k=1}^{N}[1-\cos(\phi_k-\phi_{k+1})] , \quad (12)$$

where the periodic boundary condition $\phi_{N+1}\equiv\phi_1$ is assumed. We consider the low-temperature limit and use the harmonic approximation

$$\cos(\phi_k-\phi_{k+1})\approx 1-\tfrac{1}{2}(\phi_k-\phi_{k+1})^2 .$$

We assume further that N is large and introduce a variable $\theta\equiv 2\pi k/N$ which becomes continuous in the limit $N\to\infty$. In this continuum approximation, the Euclidean action is given by

$$S_E=\int_0^\beta d\tau\int_0^{2\pi}d\theta\left[\frac{NM}{4\pi}\left[\frac{\partial\phi}{\partial\tau}\right]^2-if\frac{N}{2\pi}\left[\frac{\partial\phi}{\partial\tau}\right] +\frac{1}{2}V_0\left[\frac{2\pi}{N}\right]\left[\frac{\partial\phi}{\partial\theta}\right]^2\right] , \quad (13)$$

with the boundary conditions

$$\begin{cases}\phi(\tau,0)-\phi(\tau,2\pi)=2\pi n ,\\ \phi(\tau,\theta)-\phi(\tau+\beta,\theta)=2\pi m ,\end{cases}$$

where, in the low-temperature limit ($\beta V_0\to\infty$), m is uniform for all junctions and independent of θ. The corresponding classical solution of $\delta S_E=0$ is then

$$\phi(\tau,\theta)=-n\theta-(2\pi m/\beta)\tau , \quad (14)$$

leading to the partition function of period $1/N$:

$$Z=\vartheta_3(Nf,e^{-2\pi^2NM/\beta})\vartheta_3(0,e^{-N/2\beta V_0}) . \quad (15)$$

It is of interest to note that the above dependence on βE_C and βV_0 indicates the existence of duality in the system. For example, in the limit $\beta/NM\equiv 2\beta E_C/N\to 0$, only the Josephson coupling energy is relevant, and the N-junction partition function in Eq. (15) reduces to

$$Z=\vartheta_3(0,e^{-N/2\beta V_0}) ,$$

which has the same form as that of a free particle in the absence of flux, as given by Eq. (4). Thus the strong-coupling limit ($\alpha\equiv E_C/V_0\to 0$) of the ring of N ($\gg 1$) junctions corresponds to the weak-coupling limit ($\alpha\to\infty$) of a single junction, which reflects the duality between the two limits of the system.

To establish the duality, which is valid in the low-temperature limit, we consider a ring of N junctions in an external magnetic field

$$H=\frac{1}{2C}\sum_{k=1}^{N}(q_k-Q)^2 +V_0\sum_{k=1}^{N}[1-\cos(\phi_k-\phi_{k+1}-A_{k,k+1})] , \quad (16)$$

with gauge charge Q on each island. The bond angle $A_{k,k+1}$ is given by the line integral of the vector potential due to the applied magnetic field

$$A_{k,k+1}\equiv\frac{2\pi}{(\Phi_0/2)}\int_k^{k+1}\mathbf{A}\cdot d\mathbf{l}=\frac{4\pi\tilde f}{N} ,$$

where $\Phi_0/2\equiv hc/2e$ is the flux quantum for a Cooper pair and $\tilde f\equiv\Phi/\Phi_0$ is the enclosed flux in units of Φ_0.

In the weak-coupling *and* low-temperature limit ($\alpha\to\infty$ and $\beta V_0\to\infty$), the system behaves like a single particle and leads to the partition function

$$Z\approx\vartheta_3(Nf,e^{-2\pi^2NM/\beta}) ,$$

which is that of a particle of mass NM in the presence of flux Nf. [Compare with Eq. (15).] Thus in this limit, the Hamiltonian in Eq. (16) reduces to that of a particle on a circle.[10]

Conversely, in the strong-coupling limit, the Josephson energy is dominant and the N-junction system described by the Hamiltonian (16) with the charging energy neglected can be mapped onto a tight-binding particle on a circle of N sites.[11] The energy eigenvalues of either system are given by

$$E_k = -2\Delta \cos[(2\pi/N)(k+f)] \,, \tag{17}$$

where $\Delta \equiv NV_0/2$ is the hopping energy in the tight-binding system. For sufficiently large N, Eq. (17) reduces to $E_k \approx 4\pi^2\Delta(k+\tilde{f})^2/N^2$, and the tight-binding particle becomes essentially equivalent to a free particle on a circle of mass $N^2/4\pi^2E$, which can be recognized by comparing eigenvalues of the two systems. Hence in the limit $\alpha \to 0$, the system again reduces to a particle on a circle, of mass $N/4\pi^2V_0$ and in the presence of flux $\tilde{f}$. Therefore, both the $\alpha \to \infty$ limit and the $\alpha \to 0$ limit of the N-junction system correspond to a particle on a circle, and correspondingly, they map onto each other, which is exact in the large-size ($N \to \infty$), zero-temperature ($T \to 0$) limit. This establishes the duality with $f \leftrightarrow \tilde{f}/N$ and $E_C \leftrightarrow 2\pi^2 V_0$ between the two limits of the N-junction system.

In summary, we have investigated systems of Josephson junctions with gauge charge induced by external sources, via the instanton formalism. The partition functions are shown to display oscillating behavior as functions of the gauge charge. Effects of weak dissipation have been also considered and found to hardly change the general features for two-junction systems. In a single-junction system, on the other hand, Ohmic dissipation results in the destruction of the periodicity. The duality in the system, which is reflected by the limiting forms of the partition function, has been also pointed out. In particular, the strong-coupling and weak-coupling limits of a ring of N junctions map onto each other.

This work was supported in part by the Basic Science Research Institute Program, Ministry of Education of Korea, and in part by the SNU-Daewoo Research Fund. The Center for Theoretical Physics is funded by the Korea Science and Engineering Foundation.

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