Defect motions and smearing of Shapiro steps in Josephson-junction ladders under magnetic frustration

Beom Jun Kim, Seunghwan Kim, and Sung Jong Lee
Center for Theoretical Physics and Department of Physics, Seoul National University, Seoul 151-742, Korea
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We present simulation results on the dynamics of 1D Josephson ladder arrays at zero temperature in the presence of uniform magnetic fields when dc plus ac currents are applied. For a frustration $f = p/q$, the dynamics of the array can be described by the reduced equations for only q variables, if the initial configuration is assumed to be invariant under the q-lattice translation. When dc plus ac currents are injected, fractional Shapiro steps are found at time-averaged voltage $\langle V \rangle = (n/q)(\hbar\omega/2e)$ with n an integer and ω the external driving frequency. If the ladder array is wound into an *annular* geometry, we can have defects in the vortex configuration depending on the initial random-phase configuration which cannot evolve into q-periodic states, and these defects are shown to smear the Shapiro steps. The dynamic resistance on the smeared Shapiro step is proportional to the number density of the defects.

Josephson-junction arrays[1] (JJA's) under external current driving are typical examples of coupled nonlinear dynamical systems that exhibit diverse phenomena including hysteresis, coherent mode locking,[2] and chaos.[3] In the case of two-dimensional (2D) JJA's under a uniform magnetic field $\mathbf{B}$, frustration plays important roles in the statics and the dynamics of the arrays,[4] where physical quantities such as critical current and critical temperature show a sensitive dependence on the rationality of the magnetic frustration $f = p/q$, which is the flux per unit plaquette $\Phi \equiv B_0 a^2$ (a is the lattice spacing and we set $a = 1$ hereafter) in units of the flux quantum $\Phi_0 \equiv hc/2e$. Under the dc+ac current driving, time-averaged current-voltage (IV) characteristic curves show the so-called "fractional giant Shapiro steps" at $\langle V \rangle = (n/q)(N\hbar\omega/2e)$, where n is an interger, ω the angular frequency of the driving current, and N the number of junctions along the direction of the current driving.[2]

Here, in this paper, we consider the dynamics of a simpler system, i.e., 1D ladder arrays of parallel-coupled Josephson junctions where the superconducting islands form $2 \times N$ square arrays as shown in Fig. 1 under a uniform magnetic field $\mathbf{B} = B_0\hat{z}$, when an external dc *plus* ac current $I(t) \equiv I_d + I_a \sin\omega t$ (where I_d and I_a denote the dc and ac components of the driving current, respectively) is applied uniformly. Compared with 2D arrays, this model of one-dimensional nature is relatively simple but it still exhibits the static and dynamic properties of sensitive dependences on the rationality of the frustration f. And the model has some analogy to a 1D model of a charge density wave (CDW) system with commensurate or incommensurate pinning potential. It is also closely related to a long Josephson junction with periodic pinning potential and external magnetic field. Under external current driving and magnetic field, the physics of a *parallel* Josephson ladder is quite different from that of the usual *serial* Josephson ladders in that the motion of induced vortices along the ladder plays important roles in the parallel ladder array, while those induced vortices cannot move along the ladder in the case of serial Josephson ladders.

The static and dynamic behavior of the ladder array may be analyzed conveniently in terms of the vortex configuration of the equilibrium ground states, where in the case of frustration f, the vorticity (or vortex charge) for a given plaquette acquires the value of f or $f - 1$ depending on the sign of the current circulation around that plaquette. For a rational frustration $f = p/q$, the ground state configuration is periodic in space with a period of q lattice units[5] (q-periodic states). In the example of $f = 1/2$, we get a simple ground state configuration with positive and negative vortices alternating their positions.

If we assume that the q periodicity is retained in the dynamic situation, we can obtain a reduced set of dynamic equations based on q variables only, solutions of which necessarily would represent only those states with zero net vorticity (vortex charge). However, initial states with random-phase configurations may have nonvanishing net vorticity, which is closely related to the fact that vortices are interacting with each other via a potential of an exponentially decreasing form[6] in a 1D ladder array. In the case of annular ladder arrays, the net vorticity is conserved in the presence of the external current and therefore the states do not develop into q-periodic states, but rather into states with a finite density of defects in the vortex lattice due to the nonvanishing net vorticity.

A defect in the vortex lattice represents in general a domain boundary between different degenerate ground state configurations. In the example of $f = 1/2$, these domain boundaries are local vortex configurations such as $(++)$ or $(--)$ configurations, which may be called a *positive* and a *negative* defect, respectively. Simulations show that these defects themselves form a lattice with almost uniform spacing and move rigidly under the influence of external currents and that the Shapiro step structure is modified due to the additional linear contribution coming from the motion of the defects, with the dynamic resistance at the center of the smeared Shapiro step being proportional to the number density of the defects.

Figure 1 shows a ladder array of superconducting islands where nearest-neighboring islands form a Josephson junction. All horizontal junctions (we call these "main" junctions) are assumed to have uniform values of critical current I_{cx}, and all the vertical junctions are assumed to have critical current I_{cy}. The junction resistances in both directions are assumed to be the same, i.e., $R_x = R_y \equiv R$. It is worthy to note that I_{cy} controls the coupling between the neighboring main junctions.

Applying the current conservation law at each node, we get the following equations for the superconducting phases $\phi_{1,k}$, and $\phi_{2,k}$ ($k = 1, \ldots, N$, with N being the number of parallel junctions), on the kth islands of the left side and the right side columns, respectively:

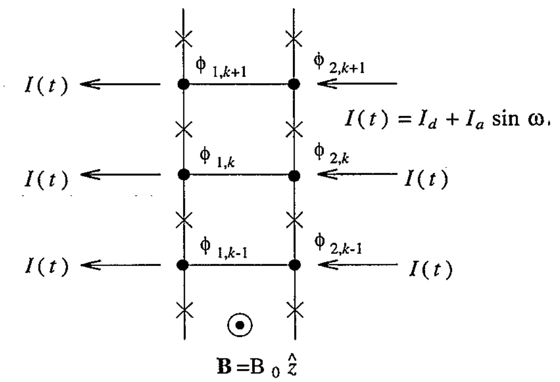

FIG. 1. Schematic picture of a ladder array of Josephson junctions in the presence of a uniform magnetic field $\mathbf{B} = B_0\hat{z}$ driven by uniform external currents $I(t) = I_d + I_a \sin\omega t$. [(×) and (•) represent a Josephson junction and a superconducting grain, respectively.]

$$3\dot{\phi}_{1,k} - \dot{\phi}_{2,k} - \dot{\phi}_{1,k+1} - \dot{\phi}_{1,k-1} = -I(t) + \sin(\phi_{2,k} - \phi_{1,k} - A_k) + I_{cy}\sin(\phi_{1,k+1} - \phi_{1,k}) + I_{cy}\sin(\phi_{1,k-1} - \phi_{1,k}) \quad \text{(1a)}$$

and

$$3\dot{\phi}_{2,k} - \dot{\phi}_{1,k} - \dot{\phi}_{2,k+1} - \dot{\phi}_{2,k-1} = I(t) - \sin(\phi_{2,k} - \phi_{1,k} - A_k) - I_{cy}\sin(\phi_{2,k} - \phi_{2,k+1}) - I_{cy}\sin(\phi_{2,k} - \phi_{2,k-1}), \quad \text{(1b)}$$

where $\dot{\phi} \equiv d\phi/dt$ and $A_k \equiv (2e/\hbar c)\int_2^1 \mathbf{A}\cdot d\mathbf{l} = 2\pi f k$ is the magnetic bond angle with the gauge $\mathbf{A} = -B_0 y\hat{x}$. Here, t is in units of $\tau \equiv \hbar/2eRI_{cx}$, and the currents are in units of I_{cx}. In order to investigate the dynamics of annular ladder arrays, we can simply impose a periodic boundary condition along the y direction (perpendicular to the external currents). The control parameter I_{cy} is important for the dynamics of the array: The limit of $I_{cy} \to 0$ corresponds to the decoupling of all the horizontal junctions (the weak coupling limit), while the other limit of $I_{cy} \to \infty$ obviously corresponds to the strong coupling limit. There are $2N$ equations for $2N$ phases $\phi_{1,k}$ and $\phi_{2,k}$ ($k = 1, \ldots, N$). Among them, one equation is redundant due to the overall U(1) phase rotation symmetry. So one of the phases can be fixed arbitrarily and we can solve the $2N - 1$ remaining equations.

As in the case of 2D arrays,[7] we may assume (motivated by the q-periodic nature of the ground state configurations) that the dynamic state also retains q periodicity:

$$\phi_{1,k+q} = \phi_{1,k},$$
$$\phi_{2,k+q} = \phi_{2,k}. \quad (2)$$

Furthermore, we observe that the condition of the zero net current along the vertical direction can be satisfied by assuming (this is a sufficient condition but not a necessary one)

$$\phi_{1,k+1} - \phi_{1,k} = \phi_{2,k} - \phi_{2,k+1}. \quad (3)$$

Based on the above assumptions, we can reduce the equations of motion for $2N - 1$ phases in terms of only the q phase differences $\theta_i \equiv \phi_{2,i} - \phi_{1,i}$:

$$2\dot{\theta}_i - \frac{1}{2}\left(\dot{\theta}_{i+1} + \dot{\theta}_{i-1}\right) = I(t) - \sin(\theta_i - A_i) - I_{cy}\sin\left(\frac{\theta_i - \theta_{i+1}}{2}\right) + I_{cy}\sin\left(\frac{\theta_i - \theta_{i-1}}{2}\right), \qquad i = 1, \ldots, q. \quad (4)$$

In the example of $f = 1/2$, the reduced equations are

$$\dot{\psi}_1 = I(t) - \sin\psi_2\cos\psi_1,$$
$$3\dot{\psi}_2 = -2I_{cy}\sin\psi_2 - \cos\psi_2\sin\psi_1, \quad (5)$$

where ψ_1 and ψ_2 are defined by $\psi_1 \equiv (\theta_1 + \theta_2)/2$ and $\psi_2 \equiv (\theta_1 - \theta_2)/2$. When dc currents [$I(t) = I_d$] are injected, there exists a threshold current I_c above which the time-averaged voltage becomes nonzero. By taking $\dot{\psi}_1 = \dot{\psi}_2 = 0$ and calculating the maximum possible value of the external current for the above equations to have solutions, we obtain the dc critical currents $I_c(f = 1/2, I_{cy})$ for the special case of $f = 1/2$, which can be expressed in a closed form

$$I_c(f = 1/2, I_{cy}) = \sqrt{4I_{cy}^2 + 1} - 2I_{cy} \quad (6)$$

in units of I_{cx}. For the special case of $I_{cy} = 1$, we obtain $I_c \approx 0.236$ from Eq. (6), which is smaller than the value for the 2D array ($I_c \approx 0.414$).[8]

We have carried out direct integrations of Eqs. (1a) and (1b) using the second-order Runge-Kutta method with the integration time interval of $\delta t = 0.05\tau$. Simulations were performed starting from both *uniform* and *random* initial phases. By rewriting Eqs. (1a) and (1b) in terms of $\theta_k \equiv \phi_{2,k} - \phi_{1,k}$ and $\zeta_k \equiv \phi_{2,k} + \phi_{1,k}$ we can decouple the whole set of equations into two sets of equations, one of which involves $\dot{\theta}_i$'s only and the other $\dot{\zeta}_i$'s only. Using this form, the tridiagonal matrix algorithm can be employed, which makes the numerical integration very efficient.[9]

First, we calculate numerically the critical current vs frustration f for different values of I_{cy}. (See Fig. 2.) We can see that as I_{cy} increases, the critical current of the array decreases. For a fixed value of I_{cy}, the critical current decreases monotonically in f at first, and then shows spiky variations near low-order rational values of $f = p/q$ due to the commensuration effects.

Figure 3(a) shows time-averaged *IV* characteristics when dc *plus* ac currents are applied, for $f = 1/2$ and $f = 1/3$, obtained from full-array simulations and those from the reduced dynamic equations (4). Fractional Shapiro steps are clearly present at $\langle V\rangle = (n/q)(\hbar\omega/2e)$. Figure 3(b) shows the effect of changing I_{cy} on the *IV* characteristics, especially on the behavior of the Shapiro step widths. Step widths for integer Shapiro steps (corresponding to n/q integer) behave differently from fractional step widths (corresponding to n/q noninteger). That is, fractional step widths vanish in the limit of $I_{cy} \to 0$, and increase as I_{cy} increases up to some maximum value and then decrease as I_{cy} further increases. This can be easily understood since in the limit of $I_{cy} \to 0$ all horizontal junctions are decoupled from one another and only integer Shapiro steps will be observed as in the case of a single resistively shunted junction (RSJ) model. As $I_{cy} \to \infty$, the system exhibits purely Ohmic behavior, leading to a straight line in the *IV* characteristic. On the other hand, integer step widths in general decrease monotonically as I_{cy} increases for some values of I_{cy}.

In the case of the annular ladder array the dynamics shows a peculiar feature which is not realized in ladder arrays with free boundary conditions. Namely, there exist many stable dynamic solutions other than the q-periodic ones. And these solutions cannot be described by the re-

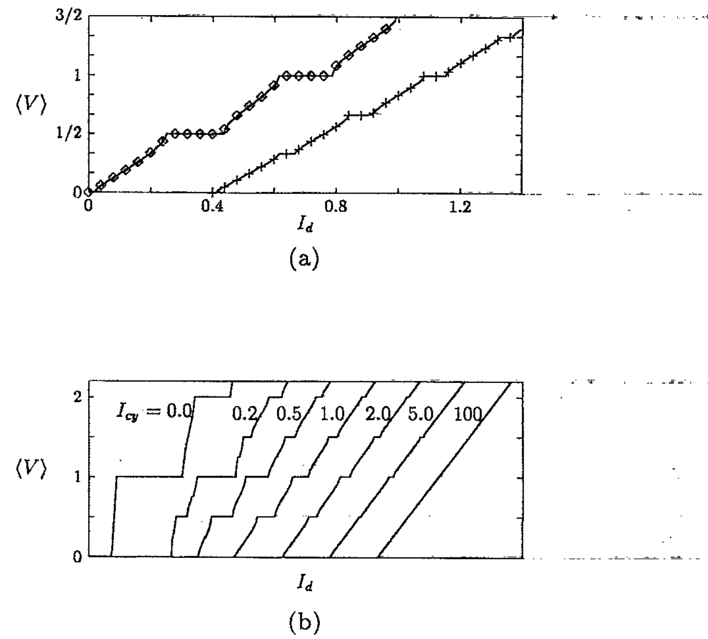

FIG. 3. (a) *IV* characteristics of ladder arrays driven by external currents $I(t) = I_d + I_a \sin\omega t$ with $I_a = 1.0$ and $\omega = 2\pi/10$ using periodic boundary conditions and q-periodic initial conditions. The time-averaged voltage drop $\langle V\rangle$ along the direction of currents is plotted in units of $\hbar\omega/2e$ from the full array simulations for $f = 1/2$ ($\diamond$) with $N = 64$ and for $f = 1/3$ (+) with $N = 81$. The thin line and the thick line are from the integration of the reduced dynamic equations for $f = 1/2$ and for $f = 1/3$ respectively. (b) *IV* characteristics from the reduced dynamic equations for $f = 1/2$ with various values of I_{cy}. ($I_a = 1.0$, $\omega = 2\pi/10$, and $\langle V\rangle$ is in units of $\hbar\omega/2e$.)

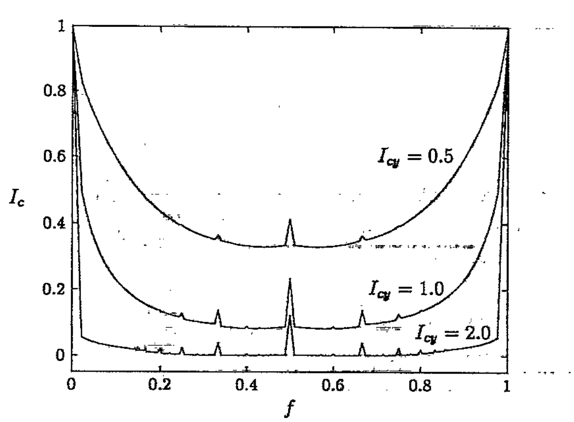

FIG. 2. Critical currents I_c vs frustration f, for various values of I_{cy}.

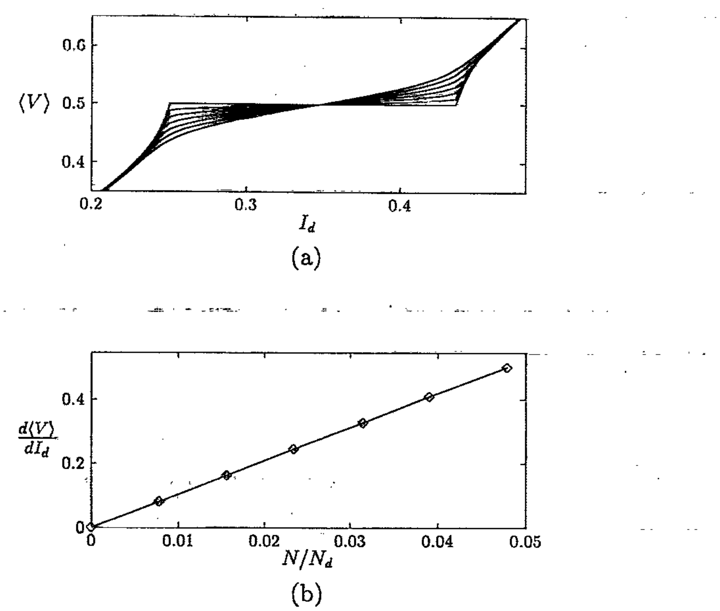

FIG. 4. (a) *IV* characteristics of annular ladder array with size $N = 256$ near 1/2 fractional Shapiro steps starting from the various random initial conditions. ($I_a = 1.0$, $I_{cy} = 1.0$, $\omega = 2\pi/10$, and $\langle V\rangle$ is in units of $\hbar\omega/2e$.) (b) Dynamic resistance ($\diamond$) on the smeared Shapiro step vs defect number density calculated from (a); solid line shows a linear fit.

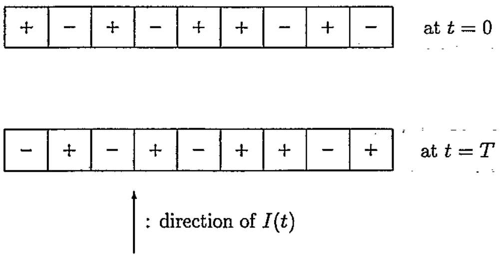

FIG. 5. Vortex configuration for a ladder array of nine plaquettes with $f = 1/2$: One positive defect (++) is shown. The vortex lattice moves one lattice unit distance per period T of external currents.

duced equations. Simulations show that these correspond to phase configurations with a finite density of defects in the periodic vortex lattice structure. In the case of free boundary conditions, these defects move along the ladder under the influence of the Lorentz force due to the external driving current and disappear through the *free* boundary of the array. On the other hand, in the case of annular (periodic) array configurations, these defects cannot disappear out of the array and they simply undergo periodic translation motions around the annular ladder array.

We monitored the time-averaged IV characteristics of annular arrays under a dc *plus* ac current driving for $f = 1/2$, using various *random* initial configurations [see Fig. 4(a)]. We can see that the Shapiro steps are not perfect, but have some finite slopes (finite dynamic resistance). We checked the phase configurations and the vortex configurations for each case of the IV curves and it was found that the dynamic resistance is proportional to the number density of the defects in the vortex lattice [Fig. 4(b)]. That is, the dynamic resistance on the smeared steps is

$$\frac{d\langle V\rangle}{dI_{\rm dc}} \approx c\frac{N_d}{N}, \tag{7}$$

where c is a constant and N_d is the net number of defects in steady states.

In order to explain these results, let us consider, for example, a vortex configuration for the nine plaquettes shown in Fig. 5. Note that this is not a q-periodic state due to the extra defect. If, in this case, the whole vortex lattice moves one unit lattice distance per period $T = 2\pi/\omega$, then it is easy to see that the voltage drop becomes

$$\langle V\rangle = \left(\frac{1}{2} + \frac{1}{2}\frac{1}{9}\right)\frac{\hbar\omega}{2e}, \tag{8}$$

where the extra term $\frac{1}{2}\frac{1}{9}\hbar\omega/2e$ comes from the motion of the extra vortex constituting the defect. We can generalize to an arbitrary density of defects moving with velocity $v(t)$ and obtain the following equation near the center of the nth Shapiro steps:

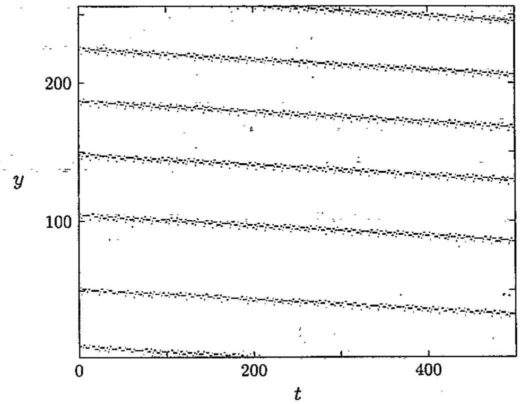

FIG. 6. Positions of the positive defects (y) vs time (t) near the center of the 1/2 Shapiro step at $I_d = 0.33$ for the same system as in Fig. 4. Separations between the vortices do not change in time. ($N = 256$, $N_d = 6$, and t is in units of τ.)

$$\frac{\langle V\rangle}{\hbar\omega/2e} = n\frac{1}{2} + \frac{T}{2}\langle j(t)\rangle, \qquad n = 1, 2, \ldots, \tag{9}$$

where $j(t) \equiv \rho v(t)$ is the current density of defects with $\rho \equiv \pm N_d/N$ being the defect *charge* density (+ is for positive defects and − for negative ones). We varied ρ and T, and all the simulation results confirmed Eq. (9).

Figure 6 shows the defect motions in time for an $N = 256$ annular ladder array where the small dots denote the positions of the defects in the ladder array. We can see that the defects are spaced at approximately regular intervals and move with constant average speed. Note that the array is wound into an annular configuration and the defect disappearing through one end actually reappears at the other end of the array. We also calculated numerically the relation between $\langle v(t)\rangle$ and the dc component of the external current I_d, and found that, near the center of the $n/q = 1/2$ fractional Shapiro step, $\langle v(t)\rangle = 2.12I_d - 0.736$. In combination with Eq. (9), this leads to $d\langle V\rangle/dI_d \propto \rho$, thus arriving at Eq. (7).

In summary, we presented simulation results on the dynamics of a 1D annular array of parallel Josephson junctions under external driving currents. We observe, through simulations, an interesting uniform translational motion of defects forming themselves an almost rigid 1D lattice around the annular ladder array, which is manifested in the IV characteristics by smearing the Shapiro steps in proportion to the density of defects. In experimental situations, it would be necessary to devise an ingeneous method to apply uniform ac current to annular ladder arrays at very low temperature. A similar phenomenon may be possible in a 1D charge density wave system or long Josephson junction under a commensurate pinning potential.

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